Electricity and Magnetism, Exam 1, 17/02/2017

17 questions

> This is a multiple-choice exam. Write your name and student number on the answer sheet. Clearly mark the answer of your choice on the answer sheet. Only a single answer is correct for every question. The score will be corrected for guessing. Use of a (graphing) calculator is allowed. You may make use of the formula sheet. The same notation is used as in the book, i.e. a bold-face $\mathbf{A}$ is a vector, T is a scalar.

For questions 1-4, consider the vectors $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + 2\hat{\mathbf{z}}$ and $\mathbf{B} = 6\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$.

1. $\mathbf{A} \cdot \mathbf{B} =$
   A. $6\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$
   B. 8
   C. 12
   D. $8\,\hat{\mathbf{z}}$

2. $\mathbf{A} \times \mathbf{B} =$
   A. $-2\hat{\mathbf{x}} + 10\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$
   B. $-6\hat{\mathbf{x}} + 10\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$
   C. $6\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$
   D. $2\hat{\mathbf{x}} - 10\hat{\mathbf{y}} - 4\hat{\mathbf{z}}$

3. $\mathbf{A} \times (\mathbf{A} + \mathbf{B}) =$
   A. $-2\hat{\mathbf{x}} + 10\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$
   B. $7\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$
   C. $-6\hat{\mathbf{x}} + 10\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$
   D. $2\hat{\mathbf{x}} - 10\hat{\mathbf{y}} - 8\hat{\mathbf{z}}$

4. $(\mathbf{A} \times \mathbf{A}) + \mathbf{B} =$
   A. $-2\hat{\mathbf{x}} + 10\hat{\mathbf{y}} + 8\hat{\mathbf{z}}$
   B. $7\hat{\mathbf{x}} + 1\hat{\mathbf{y}} + 4\hat{\mathbf{z}}$
   C. 0
   D. $6\hat{\mathbf{x}} + 2\hat{\mathbf{y}} + 2\hat{\mathbf{z}}$

   For questions 5 and 6, consider the vectors $\mathbf{C} = y\hat{\mathbf{x}} - x\hat{\mathbf{y}} + 2x\hat{\mathbf{z}}$ and $\mathbf{D} = 6x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - z\hat{\mathbf{z}}$.

5. $\boldsymbol{\nabla} \times \mathbf{C} =$
   A. 0
   B. $-2\hat{\mathbf{y}} - 2\hat{\mathbf{z}}$
   C. $y\hat{\mathbf{x}} - x\hat{\mathbf{y}} + 2x\hat{\mathbf{z}}$
   D. 2

6. $\boldsymbol{\nabla} \cdot \mathbf{D} =$
   A. 6
   B. $6\hat{\mathbf{y}} + \hat{\mathbf{y}} - \hat{\mathbf{z}}$
   C. $-2\hat{\mathbf{y}} - 2\hat{\mathbf{z}}$
   D. 0

7. Calculate the Laplacian $\nabla^2 T$ of the function $T = e^{-5x}\sin(4y)\cos(3z)$
   A. 6
   B. $-5e^{-5x}\sin(4y)\cos(3z) + 4e^{-5x}\cos(4y)\cos(3z) - 3e^{-5x}\sin(4y)\sin(3z)$
   C. 0
   D. $50T$

8. Consider the following statement: 'The line integral of the gradient of a scalar function is path independent'. This statement is:
   A. true for any gradient
   B. false for any gradient
   C. true or false, it depends on the scalar function
   D. meaningless

9. Suppose $\mathbf{v} = (2xz - 3y^2)\hat{\mathbf{y}} + (6yz^2)\hat{\mathbf{z}}$. Calculate $\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{a}$ for a square surface at $x = 0$ that extends from 0 to 1 in both y and z. The answer is:
   A. 1/3
   B. 2/3
   C. 2
   D. 4/3

10. Use the divergence theorem to find the value of the surface integral of the function
$$\mathbf{v} = y^2\hat{\mathbf{x}} + (3xy + z^2)\hat{\mathbf{y}} + (3yz)\hat{\mathbf{z}}$$
through a unit cube at the origin that extends from 0 to 1 in x, y and z. Which answer is correct?
   A. 1
   B. 2
   C. 3
   D. 4

11. The outward flux of a vector field through a closed surface is equal to ...
   A. the line integral of the gradient over the path along the surface
   B. the volume integral of the divergence over the region inside the surface
   C. the line integral of the curl along the edge of the surface

For the following two questions, consider the following vector function:
$$\mathbf{v} = s(3 + \sin^2\phi)\hat{\mathbf{s}} + s\sin\phi\cos\phi\hat{\boldsymbol{\phi}} + 3z\hat{\mathbf{z}}$$

12. What is the divergence of $\mathbf{v}$?
   A. 6
   B. 8
   C. 10
   D. 12

13. What is the curl of $\mathbf{v}$?
   A. $-2\sin\phi\cos\phi\hat{\mathbf{z}}$
   B. 0
   C. $2\sin\phi\cos\phi\hat{\mathbf{z}}$
   D. $3\hat{\mathbf{s}}+\hat{\boldsymbol{\phi}}$

14. $\int_0^\infty x^2\delta(x+2)dx =$
   A. 0
   B. 2
   C. 4
   D. ∞

15. $\int_{-\infty}^\infty x^2\delta(2-x)dx =$
   A. 0
   B. 2
   C. 4
   D. ∞

For the following two questions, suppose

$$\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}.$$

16. The divergence $\nabla\cdot\mathbf{v} =$
   A. 0
   B. $4\pi\delta^3(\mathbf{r})$
   C. 4π
   D. 8π

17. The integral $\oint_s \mathbf{v}\cdot d\mathbf{a}$ over the surface of a sphere, centered on $r=0$, with a radius of $2R$, is:
   A. 0
   B. $4\pi\delta^3(\mathbf{r})$
   C. 4π
   D. 8π

The End

Cartesian. $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

Spherical. $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient: $$\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl: $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian: $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

Cylindrical. $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

Gradient: $$\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$$

Divergence: $$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl: $$\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right]\hat{\mathbf{z}}$$

Laplacian: $$\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

VECTOR IDENTITIES

Triple Products

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

Product Rules

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Second Derivatives

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

FUNDAMENTAL THEOREMS

Gradient Theorem: $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

Divergence Theorem: $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

Curl Theorem: $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

BASIC EQUATIONS OF ELECTRODYNAMICS

Maxwell's Equations

In general:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{cases}$$

In matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \end{cases}$$

Auxiliary Fields

Definitions:

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

Linear media:

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \end{cases}$$

Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Energy, Momentum, and Power

Energy: $U = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$

Momentum: $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

Larmor formula: $P = \frac{\mu_0}{6\pi c} q^2 a^2$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}\,\mathrm{N/A^2}$	(permeability of free space)
$c = 3.00 \times 10^8\,\mathrm{m}/s$	(speed of light)
$e = 1.60 \times 10^{-19}\,\mathrm{C}$	(charge of the electron)
$m = 9.11 \times 10^{-31}\,\mathrm{kg}$	(mass of the electron)

SPHERICAL AND CYLINDRICAL COORDINATES

Spherical

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases}$$

Cylindrical

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$

$$\begin{cases} s = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases}$$